The Fibonacci's zeta function. Mathematical connections with some sectors of String Theory

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Abstract

In this paper we have described, in the **Section 1**, the Fibonacci's zeta function and the Euler-Mascheroni constant and in the **Section 2**, we have described some sectors of the string theory: zeta strings, zeta nonlocal scalar fields and some Lagrangians with zeta Function nonlocality. In conclusion, in the **Section 3**, we have described some possible mathematical connections.

1. The Fibonacci's zeta function [1], [2], [3], [4], [5], [6]

The Fibonacci's zeta function is defined as:

$$F(s) = \sum_{k=1}^{\infty} \frac{1}{Fib(k)^s} \qquad (1)$$

where s=a+jb.

<u>*The logarithm of the Fibonacci's zeta and the Fiborial*</u>

Let F(s) the Fibonacci's zeta, then is:

$$-\frac{\ln F(s)}{s} = \ln(Fib\#), \; dove\ Fib\# = \prod_{k=1}^{\infty} Fib(k) \qquad (2)$$

We have that Fib# is the Fiborial , i.e. the product to the infinity of the Fibonacci's numbers.

We can write the eq. (1) also as follows:

$$F(s) = 1^{-s} + 1^{-s} + 2^{-s} + 3^{-s} + 5^{-s} + 8^{-s} + \ldots$$

$$F(s) = e^{-s\ln 1} + e^{-s\ln 1} + e^{-s\ln 2} + e^{-s\ln 3} + e^{-s\ln 5} + e^{-s\ln 8} + \ldots$$

$$\ln F(s) = -s(\ln 1 + \ln 1 + \ln 2 + \ln 3 + \ln 5 + \ln 8 + \ldots)$$

$$-\frac{\ln F(s)}{s} = \ln \prod_{k=1}^{\infty} Fib(k) = \ln(Fib\#)$$

where we have applied a note property of the logarithms.

<u>Generalization of the Gamma function for various series</u>

If we want to reach a possible functional equation for the Fibonacci's zeta, then we need to ask first how to express the "Fibonacci's Gamma".

We known that the Gamma function Γ (x) is an extension of factorial, by an integer k to a real number x. The Gamma function was studied by Euler (see [1]) and has a significant importance in Number Theory and in the study of the Riemann zeta.

In general we know that Γ (x) enjoys some of the following properties:

1. Γ(k + 1) = k!, k ≥ 0;
2. Γ(1) = 1, Γ(x + 1) = x Γ(x) e Γ(x) is the unique solution of this functional equation for x > 0;
3. $\Gamma(x) = \lim_{k\to\infty}(k\,k^x)/(x(x+1)\ldots(x+k)) = \lim_{k\to\infty}(k^x/x)\prod_{i=1}^{k}(i/(x+i))$, x ≠ 0, -1, -2 … (*Euler form*);
4. $1/\Gamma(x) = xe^{\gamma x}\prod_{i=1}^{\infty}(1+x/i)e^{-x/i}$ (*Weierstrass form*) where γ is *the Euler-Mascheroni constant* defined by $\gamma = \lim_{k\to\infty}\sum_{i=1}^{k}\left(\frac{1}{i}\right) - \log k$;
5. $\Gamma(x)\Gamma(-x) = -\pi/(x\sin\pi x)$ (*property of reflection*), $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$ (*property of extension*) and Γ(1/2)2 = π.

From above we can tell which should be considered in the follow, both the Euler form and the Weierstrass form.

Generalized Euler form

We consider a series $g_k > 0$, $k > 0$ and an extension $g(x) > 0$, $x > 0$ with the property that:

$g(k) = g_k$ and $l(x) = \lim_{k\to\infty} g(x+k)/g(k)$ there exists and isn't zero.

From the above property 3, we have:

$$\Gamma_g(x) = \lim_{k\to\infty} g(x)\Gamma_{g,k}(x)$$
$$= \lim_{k\to\infty} \lambda(x)\frac{g(k)^x}{g(x)}\prod_{i=1}^{k}\frac{g(i)}{g(x+i)},\ g(x+i) \neq 0, i=0,1,2,... \tag{3}$$

where $\lambda(x)$ is to establish.

Thence

$$\Gamma_{g,k}(x+1) = \frac{g(k)}{g(x+k+1)}\frac{\lambda(x+1)}{\lambda(x)}g(x)\lambda(x)\frac{g(k)^x}{g(x)}\prod_{i=1}^{k}\frac{g(i)}{g(x+i)}$$

we obtain

$$\Gamma_g(x+1) = \frac{\lambda(x+1)}{l(x+1)\lambda(x)}g(x)\Gamma_g(x)$$

If we choose $\lambda(x)$ such that

$$\frac{\lambda(x+1)}{l(x+1)\lambda(x)} = 1,$$

for example:

$$\lambda(x+1) = l(x+1)\lambda(x),$$

we obtain

$$\Gamma_g(x+1) = g(x)\Gamma_g(x) \tag{4}$$

If we take $\Gamma_g(1) = 1$, we obtain $\Gamma_g(k+1) = \prod_{i=1}^{k} g_i$.

Furthermore, from eq. (3) we have:

$$\Gamma_g(1) = 1 \Rightarrow \lim_{k\to\infty} \lambda(1)\frac{g(k)}{g(k+1)}$$
$$= 1 \Rightarrow \lambda(1) = l(1)$$

such that

$$\lambda(k) = \prod_{i=1}^{k} l(i).$$

With this restriction, λ(k) can be extended to λ(x) or to have $\Gamma_g(x)$ with additional properties.

Using the eq. (4), $\Gamma_g(x)$, x > 0 satisfy the relation

$$\ln \Gamma_g(x+1) - \ln \Gamma_g(x) = \ln g(x)$$

Example A

A simple generalization of $g_k = k$ is $g_k = ak + b$ with *a*, *b* choose such that $g_k > 0$, $k > 0$. Thence, we obtain:

$$g(x) = ax + b,\ l(x) = \lim_{k\to\infty} \frac{a(x+k)+b}{ak+b} = 1 \quad \text{and} \quad \lambda(k) = 1.$$

If we take $\lambda(x) = 1$, we obtain:

$$\Gamma_g(x) = \lim_{k\to\infty} \frac{(ak+b)^x}{(ax+b)} \prod_{i=1}^{k} \frac{ai+b}{a(x+i)+b}$$

with

$$\Gamma_g(k+1) = \prod_{i=1}^{k}(ai+b).$$

The Figure 1 concerning the product of the odd numbers.

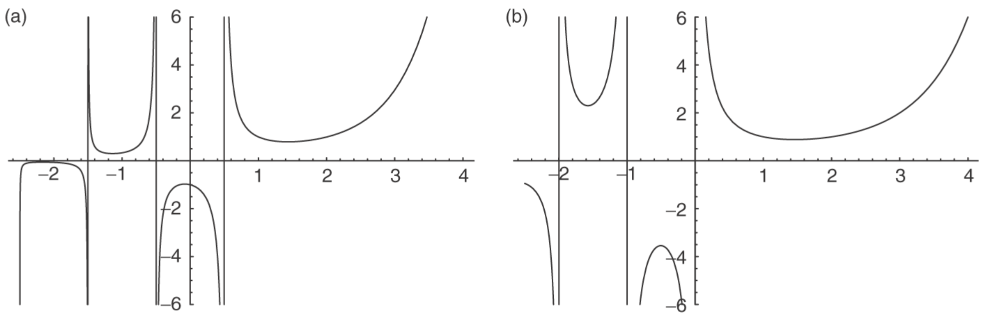

Figure 1

(a) show $\Gamma_g(x)$, an extension of the product $\Gamma_g(k+1) = \prod_{i=1}^{k}(ai+b)$ to the real numbers, for *a* = 2, *b* = -1. (b) show the comparison with the normal Gamma function $\Gamma(x)$.

Example B

For the Fibonacci's numbers is:

$$g(x) = F_x = \left(\phi^x - \phi^{-x}\cos(\pi x)\right) / \sqrt{c}$$

where $\phi = (1+\sqrt{5})/2$, i.e. the aurea ratio, and *c* = 5, and thence:

$$l(x) = \lim_{k\to\infty} \phi^{x+k}/\phi^k = \phi^x$$

and

$$\lambda(k) = \prod_{i=1}^{k} \phi^i = .$$

The most simple and effective extension is:

$$\lambda(x) = \phi^{(x+1)x/2}$$

and we obtain

$$\Gamma_F(x) = \phi^{(x+1)x/2} \lim_{k\to\infty} \frac{F_k^x}{F_x} \prod_{i=1}^{k} \frac{F_i}{F_{x+i}} \qquad (5)$$

where $\Gamma_F(k+1) = (k!)_F$.

The Figure 2 show $\Gamma_F(x)$.

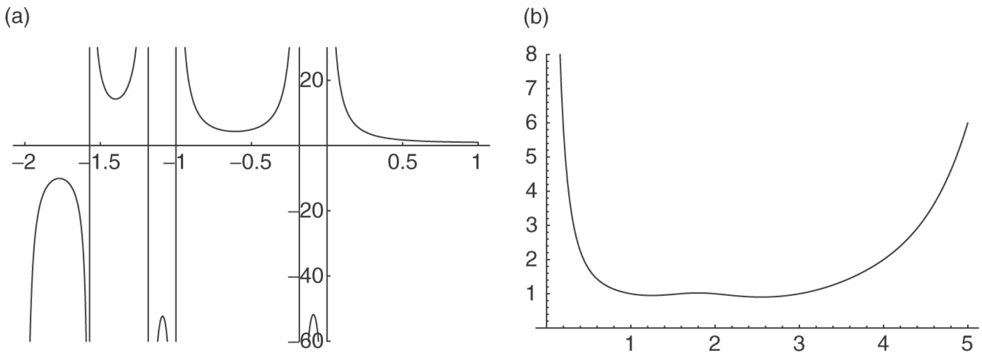

Figure 2

$\Gamma_F(x)$, an extension of the Fiborial to the real numbers with the property that $\Gamma_F(k+1) = (k!)_F$. (a) show the values for $-2 \le x \le 1$ (b) show the values for $0 \le x \le 5$.

Weierstrass generalization form

Using the eq. (3) and assuming that λ(x) and g(x) are differentiable, we obtain:

$$\frac{\Gamma'_g(x)}{\Gamma_g(x)} = \frac{\lambda'(x)}{\lambda(x)} - \frac{g'(x)}{g(x)} - \lim_{x\to\infty}\left(\sum_{i=1}^{k} \frac{g'(x+i)}{g(x+i)} - \ln g(k) \right) \qquad (6)$$

which suggests an *Euler-Mascheroni generalized constant*:

$$\gamma_g = \lim_{x\to\infty}\left(\sum_{i=1}^{k} \frac{g'(i)}{g(i)} - \ln g(k) \right) \qquad (7)$$

(we remember that the Euler-Mascheroni constant can be written also as follow

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) = \int_1^{\infty} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx, \quad (7b))$$

thence γ becomes the associated constant to the series $g_k = k$.

$$g(k)^x = \exp\left(x\left(\ln g(k) - \sum_{i=1}^{k}\frac{g'(i)}{g(i)}\right) + x\sum_{i=1}^{k}\frac{g'(i)}{g(i)}\right) = \exp\left(-x\gamma_g + x\sum_{i=1}^{k}\frac{g'(i)}{g(i)}\right)$$

Thence the *Weierstrass generalized form* (see property 4) becomes:

$$\frac{1}{\Gamma_g(x)} = \frac{g(x)}{\lambda(x)} e^{x\gamma_g} \prod_{i=1}^{\infty} \frac{g(x+i)}{g(i)} \exp^{\left(-x\frac{g'(i)}{g(i)}\right)} \quad (8)$$

While the *generalized reflection formula* becomes:

$$\Gamma_g(x)\Gamma_g(-x) = \frac{\lambda(x)\lambda(-x)}{g(x)g(-x)} \prod_{i=1}^{\infty} \frac{g(i)^2}{g(x+i)g(-x+i)} \qquad (9)$$

and since $\Gamma_g(1 - x) = g(-x)\Gamma_g(-x)$,

we obtain

$$\Gamma_g(x)\Gamma_g(1-x) = \frac{\lambda(x)\lambda(-x)}{g(x)} \prod_{i=1}^{\infty} \frac{g(i)^2}{g(x+i)g(-x+i)} \qquad (10)$$

$$\Gamma_g\left(\frac{1}{2}\right)^2 = \frac{\lambda(1/2)\lambda(-1/2)}{g(1/2)} \prod_{i=1}^{\infty} \frac{g(i)^2}{g(1/2+i)g(-1/2+i)} \qquad (11)$$

Example A

For $g_k = ak + b$, as before, the *Weierstrass form* is:

$$\frac{1}{\Gamma_g(x)} = (ax+b)e^{x\gamma_g} \prod_{i=1}^{\infty} \left(1 + \frac{ax}{ai+b}\right) e^{-xa/(ai+b)}$$

where

$$\begin{aligned}\gamma_g &= \lim_{k\to\infty}\left(\sum_{i=1}^{k}\frac{a}{ai+b} - \ln(ak+b)\right)\\ &= -a/b - \ln a - \psi_0\left(\frac{b}{a}\right)\end{aligned}$$

(ψ_0 is the *digamma function*), and the eq. (11) gives:

$$\Gamma_g\left(\frac{1}{2}\right) = \frac{1}{\sqrt{a/2+b}} \prod_{i=1}^{\infty} \frac{ai+b}{\sqrt{(ai+b)^2 - a^2/4}}$$
$$= \frac{\Gamma(b/a+1/2)}{\sqrt{a}\Gamma(b/a+1)}$$

The functional equation $\Gamma_g(x+1) = (ax+b)\Gamma_g(x)$, thence, becomes

$$\Gamma_g(k+1/2) = \prod_{i=1}^{k} (a(i-1/2)+b)\Gamma_g\left(\frac{1}{2}\right)$$
$$= (a/2-b)^k k!\Gamma_g\left(\frac{1}{2}\right) \quad \text{for } k > 0$$

Example B

For the Fibonacci's numbers the *Weierstrass form* is:

$$\frac{1}{\Gamma_F(x)} = \phi^{-(x+1)x/2} F_x e^{x\gamma_F} \prod_{i=1}^{\infty} \frac{F_{x+i}}{F_i} e^{-xF_i'/F_i} \qquad (12)$$

where $\gamma_F = 0.6676539532 \ldots$, and the eq. (11) gives

$$\Gamma_F\left(\frac{1}{2}\right) = \frac{\phi^{1/8}}{\sqrt{F_{1/2}}} \prod_{i=1}^{\infty} \frac{F_i}{\sqrt{F_{1/2+i}F_{-1/2+i}}}$$

since $F_{\pm 1/2+i} = \phi^{i\pm 1/2}/\sqrt{c}$ for i integer.

We have that:

$$\Gamma_F\left(\frac{1}{2}\right) = \left(\frac{c^2}{\phi}\right)^{1/8} \prod_{i=1}^{\infty} \left(1 - (-\phi^{-2})^i\right) = 1.7273179426\ldots$$

Alternatively, can be considered a closed form for the constant of the Fiborial (Sloane's A062073 [4])

$$\prod_{i=1}^{\infty} \left(1 - (-\phi^{-2})^i\right) = \left(\frac{\phi}{c^2}\right)^{1/8} \Gamma_F\left(\frac{1}{2}\right) \quad (=1.2267420107\ldots)$$

(If we replace $\Gamma_F(1/2)$ with $\Gamma_g(1/2)$ where $g(x) = \left(\phi^x - \phi^{-x}\cos(\pi x)\right)/\sqrt{c}$ and φ is the aurea ratio , the precedent result can give a closed form for the infinite product where $\varphi > 1$. This product is linked to the partitions functions and the q-series)

$$\Gamma_F(k+1/2)=\prod_{i=1}^{k}F_{i-1/2}\Gamma_F\left(\frac{1}{2}\right)=\frac{\phi^{k^2/2}}{c^{k/2}}\Gamma_F\left(\frac{1}{2}\right)\quad \text{for } k>0$$

<u>*Other series and generalizations*</u>

Generalized Gamma functions can be obtained also for series as: $g(x)=\sum_i c_i x^i$, $g(x)=(\sum_i c_i x^i)^{-1}$ and $g(x)=\sum_i c_i r_i^x$ etc. The Euler-Mascheroni generalized constant can also be used to introduce sequences that are well known. If $g'(i)/g(i)=f(i)$ then $g(i)=Ae^{\int f(i)di}$ (*A* constant) such that:

$$\gamma_g=\lim_{k\to\infty}\left(\sum_{i=1}^{k}f(i)-\ln g(k)\right)$$

For example, $f(i)=1/i^z$, $z>0$ from $g(x)=e^{x^{1-z}/1-z}$ with $\gamma_g=\zeta(z)$, the Riemann zeta.

$\binom{n}{m}_g=\frac{\prod_{i=1}^{n}g_i}{\prod_{i=1}^{m}g_i\prod_{i=1}^{n-m}g_i}$ can be extended to real numbers using:

$$\binom{x}{y}_g=\frac{\Gamma_g(x+1)}{\Gamma_g(y+1)\Gamma_g(x-y+1)}$$

The *Beta function* can be also generalized to:

$$B_g(x,y)=\frac{\Gamma_g(x)\Gamma_g(y)}{\Gamma_g(x+y)}$$

The logarithmic derivative, analyzed previously, generalizes the *digamma function*

$\psi_0(x)=\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)=-\gamma+\sum_{i=1}^{\infty}\frac{x-1}{i(x+i-1)}$ with $\psi_0(1)=-\gamma$. Thence:

$$\begin{aligned}\psi_{g,0}(x)&=\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma_g(x)=\frac{\Gamma_g'(x)}{\Gamma_g(x)}\\&=\frac{\lambda'(x)}{\lambda(x)}-\gamma_g+\sum_{i=1}^{\infty}\frac{g'(i)g(x+i-1)-g(i)g'(x+i-1)}{g(i)g(x+i-1)},\end{aligned}$$

$g(x+i)\neq 0$, $i=0, 1, 2, \ldots$, using the *Weierstrass form*.

This satisfies also other relations:

$$\psi_{g,0}(x+1) - \psi_{g,0}(x) = \frac{g'(x)}{g(x)}.$$

Considering

$$\psi_0(k) = -\gamma_g + H_{k-1},$$

where $H_k = \sum_{i=1}^{k} 1/i$ is the k-mo harmonic number, the *'g-harmonic' number* can be defined as $H_{gk} = \sum_{i=1}^{k} g'(i)/g(i)$.

Analogously, if we assume that g(x) is differentiable, also the properties of the *poligamma function* $\psi_{g,k}(x) = \mathrm{d}^k/\mathrm{d}x^k \psi_{g,0}(x)$ can be considered.

For example:

$$\psi_k(1) = \sum_{i=1}^{\infty} (-\ln i)^{(k+1)} = (-1)^{k+1} k! \zeta(k+1), \quad k > 0,$$

while

$$\psi_{g,k}(1) = (\ln \lambda(x))^{(k+1)}\Big|_{x=1} + \sum_{i=1}^{\infty} (-\ln g(i))^{(k+1)}$$

Thence a *'g-zeta'* can be defined as

$$\zeta_g(k) = \frac{(-1)^k}{(k-1)!} \sum_{i=1}^{\infty} (-\ln g(i))^{(k)} \quad \text{with } k>1$$

when g(i) = ai + b , we obtain

$$\zeta_g(k) = \sum_{i=1}^{\infty} \frac{1}{(i+b/a)^k}$$

that can be extended to the real numbers and becomes the *Hurwitz zeta function*:

$$\zeta(b/a+1, x) = \sum_{i=0}^{\infty} \frac{1}{(i+b/a+1)^x}$$

For the Fibonacci's zeta, thence, we can take in consideration the eq. (5) or the eq. (12).

1.1 The Euler-Mascheroni constant

The Euler-Mascheroni constant (see eq. (7b)) is a mathematical constant, used principally in number theory and in the mathematical analysis. It is defined as the limit of the difference between the truncated harmonic series and the natural logarithm:

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \ln(n) \right),$$

that we can rewrite in the following form:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = \int_1^{\infty}\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx, \quad (7b)$$

where $\lfloor x\rfloor$ is the part integer function.

We have that γ has the following approximated value $\gamma \cong 0,5772156649....$ The constant can be defined in various modes by the integrals:

$$\gamma = -\int_0^{\infty}\frac{\ln(x)}{e^x}dx = -\int_0^1 \ln\ln\left(\frac{1}{x}\right)dx = \int_0^{\infty}\left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right)e^{-x}dx = \int_0^{\infty}\frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx. \quad (7c)$$

The Euler-Mascheroni constant is related to the zeta function from the following expressions:

$$\gamma = \ln\left(\frac{4}{\pi}\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\zeta(m+1)}{2^m(m+1)}; \qquad \gamma = \lim_{s\to 1^+}\sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \frac{1}{s^n}\right) = \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right). \quad (7d)$$

The constant is also linked to the gamma function:

$$\gamma = \lim_{x\to\infty} x - \Gamma\left(\frac{1}{x}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\left\lceil\frac{n}{k}\right\rceil - \frac{n}{k}\right), \quad (7e)$$

and to the beta function:

$$\gamma = \lim_{n\to\infty}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma(n+1)n^{1+1/n}}{\Gamma\left(2+n+\frac{1}{n}\right)} - \frac{n^2}{n+1}. \quad (7f)$$

Now we describe two theorems related to the Euler-Mascheroni constant γ .

Theorem (1).

For positive integers a *we have*

$$\lim_{a\to\infty}\frac{1}{a}\int_1^a\{a/x\}dx = 1-\gamma \qquad \textit{where } \gamma \textit{ is the Euler-Mascheroni constant}.$$

We have:

$$\int_1^a \{a/x\}dx = \int_1^a (a/x)dx - \int_1^a \lfloor a/x \rfloor dx = \int_1^a (a/x)dx - \left(\int_1^{(a/a-1)} \left\lfloor \frac{a}{x} \right\rfloor dx + \int_{(a/a-1)}^{(a/a-2)} \left\lfloor \frac{a}{x} \right\rfloor dx + ..+ \int_{(a/2)}^{a} \left\lfloor \frac{a}{x} \right\rfloor dx \right) =$$

$$= \int_1^a \left(\frac{a}{x} \right) dx - \sum_{i=1}^{a-1} \left(\frac{a}{(a-i)} - \frac{a}{(a-i+1)} \right)(a-i) = a\ln(a) - \sum_{i=1}^{a} \frac{a}{i} + a .$$

Therefore,

$$\lim_{a\to\infty} \frac{1}{a} \int_1^a \{a/x\} dx = \lim_{a\to\infty} \frac{1}{a} \left(a\ln(a) - \sum_{i=1}^{a} \frac{a}{i} + a \right) = \lim_{a\to\infty} \left(\ln(a) - \sum_{i=1}^{a} \frac{1}{i} + 1 \right) =$$

$$= 1 - \lim_{a\to\infty} \left(H_a - \ln(a) \right) = 1 - \gamma \quad . \quad \text{(7g)}$$

Theorem (2).

If p_n *denotes the n*th *prime number, then we get as* $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{d(i)}{p_i} - \frac{\ln(p_n)}{n-1} = 0 \qquad \textit{where} \;\; d(i) = p_{i+1} - p_i .$$

We start by using the result of theorem (1), as $n \to \infty$

$$1 - \gamma = \frac{1}{p_n} \int_1^{p_n} \left\{ \frac{p_n}{x} \right\} dx = \frac{1}{p_n} \int_1^{p_1} \left\{ \frac{p_n}{x} \right\} dx + \frac{1}{p_n} \int_{p_1}^{p_2} \left\{ \frac{p_n}{x} \right\} dx + ... \frac{1}{p_n} \int_{p_{n-2}}^{p_{n-1}} \left(\frac{p_n}{x} \right) dx + \frac{1}{p_n} \int_{p_{n-1}}^{p_n} \left(\frac{p_n}{x} \right) dx =$$

$$= \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\{ \frac{p_n}{x} \right\} dx = \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \frac{p_n}{x} dx - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\lfloor \frac{p_n}{x} \right\rfloor dx =$$

$$= \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\lfloor \frac{p_n}{x} \right\rfloor dx . \quad \text{(7h)}$$

Now we proceed in two directions:

1)

$$1 - \gamma < \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} (p_{i+1} - p_i) \left(\frac{p_n}{p_i} - 1 \right) = \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_i} + \frac{1}{p_n} \sum_{i=0}^{n-1} (p_{i+1} - p_i) =$$

$$= \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_i} + \frac{p_n - 1}{p_n} < \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_i} + 1 = \ln(p_n) - \sum_{i=1}^{n-1} \frac{d(i)}{p_i} .$$

Hence we get

$$\sum_{i=1}^{n-1} \frac{d(i)}{p_i} < \ln(p_n) - 1 + \gamma \; .$$

2)

$$1-\gamma = \ln(p_n) - \frac{1}{p_n}\sum_{i=0}^{n-1}\int_{p_i}^{p_{i+1}}\left\lfloor \frac{p_n}{x}\right\rfloor dx > \ln(p_n) - \frac{1}{p_n}\sum_{i=0}^{n-1}(p_{i+1}-p_i)\frac{p_n}{p_{i+1}} = \ln(p_n) - \sum_{i=0}^{n-1}\frac{d(i)}{p_{i+1}} =$$

$$= \ln(p_n) - \sum_{i=1}^{n-1}\frac{d(i)}{p_{i+1}} - 1 . \quad \text{(7i)}$$

Hence we get

$$\sum_{i=1}^{n-1}\frac{d(i)}{p_{i+1}} > \ln(p_n) - 2 + \gamma \ .$$

Combining, the above two cases, we get

$$\ln(p_n) - 2 + \gamma < \sum_{i=1}^{n-1}\frac{d(i)}{p_{i+1}} < \sum_{i=1}^{n-1}\frac{d(i)}{p_i} < \ln(p_n) - 1 + \gamma \ .$$

Dividing by $n-1$ and taking limits as $n \to \infty$, we get the following result, i.e.,

$$\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{d(i)}{p_i} - \frac{\ln(p_n)}{n-1} = 0 .$$

2. Zeta Strings, Zeta Nonlocal Scalar Fields and Some Lagrangians with Zeta Function Nonlocality [7], [8], [9]

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (13)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-++\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\geq 1} C_n\mathcal{L}_n = \sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (14)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (15)$$

Employing usual expansion for the logarithmic function and definition (15) we can rewrite (14) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\,\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (16)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon, \quad (17)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (18)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (19)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (20)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (21)$$

and one can easily see trivial solution $\phi = \theta = 0$.

The exact tree-level Lagrangian of effective scalar field φ , which describes open p-adic string tachyon, is:

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (22)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (22) with p replaced by $n \in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi + \frac{1}{n+1}\phi^{n+1}\right], \quad (23)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (24)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h}\phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1}\phi^{n+1} \right] \quad (25)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h} = \prod_p \frac{1}{1-p^{-\frac{\Box}{2m^2}-h}} . \quad (26)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau , \quad \sigma > 1, \quad (27)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (27) we can rewrite (25) in the form

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\, \zeta\left(\frac{\Box}{2m^2}+h\right)\phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1}\phi^{n+1} \right] . \quad (28)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk , \quad (29)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (28) with analytic continuations of the zeta function and the power series $\sum \frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\,\zeta\left(\frac{\square}{2m^2}+h\right)\phi + AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (30)$$

where AC denotes analytic continuation.

Potential of the above zeta scalar field (30) is equal to $-L_h$ at $\square = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (31)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ -function vanishes at $h = -2,-4,-6,\dots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\square}{2m^2}+h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n, \quad (32)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n, \quad (33)$$

respectively.

Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (30): $h = 0,\ h = \pm 1,$ and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\square}{2m^2}-2\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (34)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (34).

For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\square}{2m^2} - 1\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (35)$$

where $\zeta(-1) = -\frac{1}{12}$.

The equation of motion (35) has a constant trivial solution only for $\phi(x) = 0$.

For $h = 0$, the equation of motion is

$$\zeta\left(\frac{\square}{2m^2}\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (36)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\square}{2m^2}\right) = \zeta(0) + \sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\square}{2m^2}\right)^n, \quad (37)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.

For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} + 1\right)\tilde{\phi}(k)dk = -\frac{1}{2}\ln(1-\phi)^2, \quad (38)$$

where $\zeta(1) = \infty$ gives $V_1(\phi) = \infty$.

In conclusion, for $h = 2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk = -\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw. \quad (39)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

one has trivial solution $\phi = 1$ in (39).

Now, we want to analyze the following case: $C_n = \frac{n^2-1}{n^2}$. In this case, from the Lagrangian (23), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right\}\phi + \frac{\phi^2}{1-\phi}\right]. \quad (40)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2. \quad (41)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n\pm 1$ with n =1 and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (41b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (42)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi = 0, \quad (43)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right) = 2. \quad (44)$$

From (44) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

We note that the number 2.79 is connected with $\phi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$, i.e. the "aurea" section and the "aurea" ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right) = 2{,}772542 \cong 2{,}78.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-25/7} = 2{,}618033989 + 0{,}179314566 = 2{,}79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n = \frac{n^2-1}{n^2}$, are:

$$L = \frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2} - \zeta\left(\frac{\Box}{2m^2} - 1\right) - \zeta\left(\frac{\Box}{2m^2}\right) - 1\right\}\phi + \frac{\phi^2}{2}\ln\phi^2 + \frac{\phi^2}{1-\phi}\right], \quad (45)$$

$$V(\phi) = \frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1) + \zeta(0) + 1 - \ln\phi^2 - \frac{1}{1-\phi}\right], \quad (46)$$

$$\left[\zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right) - \frac{\Box}{m^2} + 1\right]\phi = \phi\ln\phi^2 + \phi + \frac{2\phi - \phi^2}{(1-\phi)^2}, \quad (47)$$

$$\zeta\left(\frac{M^2}{2m^2} - 1\right) + \zeta\left(\frac{M^2}{2m^2}\right) = \frac{M^2}{m^2}. \quad (48)$$

In addition to many tachyon solutions, equation (48) has two solutions with positive mass: $M^2 \approx 2.67m^2$ and $M^2 \approx 4.66m^2$.

We note also here, that the numbers 2.67 and 4.66 are related to the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right) \cong 2.6798,$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \left(\frac{\sqrt{5}+1}{2}\right) + \frac{1}{2^2}\left(\frac{\sqrt{5}+1}{2}\right) \cong 4.64057.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-41/7} = 2{,}618033989 + 0{,}059693843 = 2{,}6777278;$$

$$(\Phi)^{22/7} + (\Phi)^{-30/7} = 4{,}537517342 + 0{,}1271565635 = 4{,}6646738.$$

Now, we describe the case of $C_n = \mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n) = \begin{cases} 0, & n = p^2 m \\ (-1)^k, & n = p_1 p_2 \ldots p_k, p_i \neq p_j \\ 1, & n = 1, (k = 0). \end{cases} \quad (49)$$

The corresponding Lagrangian is

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{\frac{\Box}{2m^2}}}\phi + \sum_{n=1}^{+\infty} \frac{\mu(n)}{n+1}\phi^{n+1} \right] \quad (50)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (51)$$

Now (50) can be rewritten as

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^\infty \mathcal{M}(\phi)\, d\phi \right], \quad (52)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n)\phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\Box = 0) = \frac{m^D}{g^2}\left[\frac{C_0}{2}\phi^2 (1 - \ln\phi^2) - \phi^2 - \int_0^\phi \mathcal{M}(\phi)\, d\phi \right], \quad (53)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi - \mathcal{M}(\phi) - C_0 \frac{\Box}{m^2}\phi - 2C_0 \phi \ln\phi = 0, \quad (54)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0\frac{M^2}{m^2} + 2C_0 - 1 = 0 \quad , \quad |\phi| << 1, \quad (55)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.

Now, we take the pure numbers concerning the eqs. (44) and (48). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \quad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \quad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (56)$$

3. Mathematical connections

Now we take the eqs. (5) and (12) of the **Section 1**, and the eqs. (18), (21) and (36) of **Section 2**. We obtain the following mathematical connections:

$$\Gamma_F(x) = \phi^{(x+1)x/2} \lim_{k\to\infty} \frac{F_k^x}{F_x} \prod_{i=1}^{k} \frac{F_i}{F_{x+i}} \Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k)\, dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \tilde{\theta}(k)\, dk = \sum_{n\geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} \left(\phi^{n+1} - 1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \tilde{\phi}(k)\, dk = \frac{\phi}{1-\phi}; \quad (57)$$

$$\frac{1}{\Gamma_F(x)} = \phi^{-(x+1)x/2} F_x e^{x\gamma_F} \prod_{i=1}^{\infty} \frac{F_{x+i}}{F_i} e^{-xF_i'/F_i} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \widetilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \widetilde{\theta}(k) dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \widetilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (58)$$

Now we take the eqs. (7h) and (7i) of the **Section 1**, and the eqs. (18), (21) and (36) of **Section 2**. We obtain the following mathematical connections:

$$1-\gamma = \frac{1}{p_n}\int_1^{p_n}\left\{\frac{p_n}{x}\right\}dx = \ln(p_n) - \frac{1}{p_n}\sum_{i=0}^{n-1}\int_{p_i}^{p_{i+1}}\left\lfloor\frac{p_n}{x}\right\rfloor dx \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \widetilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \widetilde{\theta}(k) dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \widetilde{\phi}(k) dk = \frac{\phi}{1-\phi}; \quad (59)$$

$$1-\gamma = \ln(p_n) - \frac{1}{p_n}\sum_{i=0}^{n-1}\int_{p_i}^{p_{i+1}}\left\lfloor\frac{p_n}{x}\right\rfloor dx = \ln(p_n) - \sum_{i=1}^{n-1}\frac{d(i)}{p_{i+1}} - 1 \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \widetilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (60)$$

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